

Numerical fractional differentiation: stability estimate and regularization

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Abstract. In this paper we prove a stability estimate for a problem of fractional differentiation. Based on the obtained stability estimate, we present a Tikhonov regularization and obtain the error estimate.

Key words. Fractional differentiation, ill-posed problems, Tikhonov regularization, stability estimate, error estimate

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1 Introduction

The basic theory and applications of fractional differential equations are covered extensively in the literature [1][2] etc. However, the topics related to ill-posed inverse problems are few, see [3][4][5].

Fractional differentiation problems arises in many contexts and have important applications in science and engineering [6][7][8]. In this paper, we consider the problem of fractional differentiation in $L^2(\mathbb{R})$. We want to compute

$$g(x) := D^\beta f(x) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_{-\infty}^x \frac{f(t)}{(x-t)^\beta} dt \quad (1.1)$$

which is called the β -th order fractional derivative of the function $f(x)$ with $\beta \in (0, 1]$, where $\Gamma(\cdot)$ is the Euler Gamma function. If there exists noise in $f(x)$, then the problem $g(x) = D^\beta f(x)$ for solving $g(x)$ is an ill-posed problem. The problem of fractional differentiation amounts to the problem of solving the integral equation of the first kind [9][10][11]

$$(A_\beta u)(x) := \frac{1}{\Gamma(\beta)} \int_{-\infty}^x \frac{g(t)}{(x-t)^{1-\beta}} dt = f(x). \quad (1.2)$$

This problem have been studied by many authors [12][13][14][15].

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Assume that

$$\|f(\cdot) - f^\delta(\cdot)\| \leq \delta, \quad (1.3)$$

where $\|\cdot\|$ denotes the L^2 -norm, and the constant $\delta > 0$ represents the noise level.

Let

$$\hat{s}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} s(x) e^{-i\xi x} dx \quad (1.4)$$

be the Fourier transform of the function $s(x) \in L^2(\mathbb{R})$. The corresponding inverse Fourier transform of the function $\hat{s}(\xi)$ is

$$s(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{s}(\xi) e^{i\xi x} d\xi. \quad (1.5)$$

In the frequency domain, we have

$$(i\xi)^\beta \hat{f}(\xi) = \hat{g}(\xi). \quad (1.6)$$

From (1.6), we can formulate the problem as an operator equation

$$\hat{A}\hat{g}(\xi) = \hat{f}(\xi), \quad (1.7)$$

where $\hat{A} = (i\xi)^{-\beta} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is a self-adjoint multiplication operator.

Since the factor $(i\xi)^\beta \rightarrow \infty$ in (1.6) as $\xi \rightarrow \infty$, it is easy to see that the ill-posedness of Problem (1.1).

However, most of the literature is devoted to the regularization method and error estimate. The conditional stability estimate is not clear. In this paper, our task is to establish a stability estimate. For error estimate of regularization, the method in this study is different from the existing literature, e.g., [13]

- In [13], the authors got the error estimate of mollification regularization by direct calculation the difference between the regularized solution and exact one.
- In this study, we use the obtained stability estimate and we only need to estimate one term.

The advantage of our method for obtaining error estimate over the method in [13] lie in:

- we can obtain stability estimate which is an important issue for study on ill-posed problems;
- fewer quantities are needed to be estimated, thus, our method is easier to obtain error estimate;

The paper is organized as follows. In Section 2, a conditional stability estimate for problem (1.1) is proved ; in Section 3, the error estimate for a Tikhonov regularization method is given.

2 A conditional stability estimate

A conditional stability estimate for ill-posed problems tells how much any two solutions differ from each other when some error exists. Since problem (1.1) is linear, we only need to derive the stability estimate on the solution near zero between the zero one.

For problem (1.1), we assume that there exists a-priori bound:

$$\|g(\cdot)\|_p \leq E, \quad (2.1)$$

where $\|\cdot\|_p$ denotes the norm of Sobolev space $H^p(\mathbb{R})$ and $p > 0$, E is a positive constant.

Thus, we can establish the stability estimate for problem (1.1).

Theorem 2.1. *Suppose that $g(x)$ is the solution of problem (1.1), and (2.1) is satisfied, then the following estimate holds :*

$$\|g(\cdot)\| \leq \|f(\cdot)\| + \|g(\cdot)\|_p^{\frac{\beta}{p+\beta}} \|f(\cdot)\|_p^{\frac{p}{p+\beta}}. \quad (2.2)$$

Proof. Since $\hat{g}(\xi) = (i\xi)^\beta \hat{f}(\xi)$, by Parseval identity we have

$$\begin{aligned} \|g(\cdot)\|^2 &= \|\hat{g}(\cdot)\|^2 = \int_{|\xi| \leq 1} |\xi|^{2\beta} |\hat{f}(\xi)|^2 d\xi \\ &+ \int_{|\xi| \geq 1} |\hat{g}(\xi)|^2 d\xi := B_1 + B_2. \end{aligned}$$

Obviously,

$$B_1 \leq \|f(\cdot)\|^2.$$

Further by Hölder inequality,

$$\begin{aligned} B_2 &= \int_{|\xi| \geq 1} |\hat{g}(\xi)|^2 d\xi \\ &= \int_{|\xi| \geq 1} [(1 + \xi^2)^p |\hat{g}(\xi)|^2]^{\frac{\beta}{p+\beta}} [(1 + \xi^2)^{-\beta} |\hat{g}(\xi)|^2]^{\frac{p}{p+\beta}} d\xi \\ &\leq \left(\int_{|\xi| \geq 1} (1 + \xi^2)^p |\hat{g}(\xi)|^2 d\xi \right)^{\frac{\beta}{p+\beta}} \cdot \left(\int_{|\xi| \geq 1} (1 + \xi^2)^{-\beta} |\hat{g}(\xi)|^2 d\xi \right)^{\frac{p}{p+\beta}} \\ &\leq \|g(\cdot)\|_p^{\frac{2\beta}{p+\beta}} \cdot \left(\int_{|\xi| \geq 1} (1 + \xi^2)^{-\beta} |(i\xi)^\beta \hat{f}(\xi)|^2 d\xi \right)^{\frac{p}{p+\beta}} \\ &\leq \|g(\cdot)\|_p^{\frac{2\beta}{p+\beta}} \cdot \left(\sup_{|\xi| \geq 1} \frac{\xi^{2\beta}}{(1 + \xi^2)^\beta} \int_{-\infty}^{+\infty} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{p}{p+\beta}} \\ &\leq \|g(\cdot)\|_p^{\frac{2\beta}{p+\beta}} \cdot \|f(\cdot)\|_p^{\frac{2p}{p+\beta}}. \end{aligned}$$

Hence we have (2.2). □

Remark 2.1. *For given two functions $f_1(\cdot)$ and $f_2(\cdot)$, let $g_1(\cdot)$ and $g_2(\cdot)$ be the corresponding solutions, respectively, then*

$$\|g_1(\cdot) - g_2(\cdot)\| \leq \|f_1(\cdot) - f_2(\cdot)\| + \|g_1(\cdot) - g_2(\cdot)\|_p^{\frac{\beta}{p+\beta}} \|f_1(\cdot) - f_2(\cdot)\|_p^{\frac{p}{p+\beta}}. \quad (2.3)$$

3 Tikhonov regularization and error estimate

Now we focus on the regularization problem. Assume that the noisy data $f_\delta(\cdot)$ satisfies

$$\|f(\cdot) - f_\delta(\cdot)\| \leq \delta, \quad (3.1)$$

where $\delta > 0$ is the noise level.

Here we use Tikhonov regularization method [16]. The method consists of looking for the solution for problem (1.1), and minimizes the quadratic functional:

$$\|f(\cdot) - f_\delta\|^2 + \alpha \|g(\cdot)\|_p^2, \quad (3.2)$$

where $\alpha = \delta^2/E^2$. By Parseval's identity and (1.6) applied to $f(\cdot)$, the variational problem becomes

$$\|(i\xi)^{-\beta} \hat{g}(\cdot) - f_\delta(\cdot)\|^2 + \alpha \|(1 + \xi^2)^{p/2} \hat{g}(\cdot)\|^2. \quad (3.3)$$

Let $\hat{g}_\delta^\alpha(\cdot)$ be the solution of above problem, the it satisfies the Euler equation

$$(|\xi|^{-2\beta} + \alpha(1 + \xi^2)^p) \hat{g}_\delta^\alpha(\xi) = (i\xi)^{-\beta} \hat{f}_\delta(\xi). \quad (3.4)$$

The Tikhonov regularization solution $\hat{g}_\delta^\alpha(\cdot)$ in frequency domain can be given

$$\hat{g}_\delta^\alpha(\xi) = \frac{(i\xi)^\beta}{1 + \alpha(1 + \xi^2)^p |\xi|^{2\beta}} \hat{f}_\delta(\xi). \quad (3.5)$$

Now we will use the stability estimate (2.3) to derive the error estimate between the regularization solution and the exact one. Via Parseval's identity, we have

$$\begin{aligned} \|\hat{g}_\delta^\alpha(\cdot) - \hat{g}(\cdot)\|_p &= \left(\int_{-\infty}^{\infty} (1 + \xi^2)^p \left| \frac{(i\xi)^\beta}{1 + \alpha(1 + \xi^2)^p |\xi|^{2\beta}} \hat{f}_\delta - (i\xi)^\beta \hat{f} \right|^2 d\xi \right)^{1/2} \\ &\leq \left(\int_{-\infty}^{\infty} (1 + \xi^2)^p \left| \frac{(i\xi)^\beta}{1 + \alpha(1 + \xi^2)^p |\xi|^{2\beta}} \hat{f}_\delta - \frac{(i\xi)^\beta}{1 + \alpha(1 + \xi^2)^p |\xi|^{2\beta}} \hat{f} \right|^2 d\xi \right)^{1/2} \\ &\quad + \left(\int_{-\infty}^{\infty} (1 + \xi^2)^p \left| \frac{(i\xi)^\beta}{1 + \alpha(1 + \xi^2)^p |\xi|^{2\beta}} \hat{f} - (i\xi)^\beta \hat{f} \right|^2 d\xi \right)^{1/2} \\ &\leq \sup_{\xi \in R} \left| \frac{(1 + \xi^2)^{p/2} (i\xi)^\beta}{1 + \alpha(1 + \xi^2)^p |\xi|^{2\beta}} \right| \|\hat{f}_\delta - \hat{f}\| + \sup_{\xi \in R} \left| \frac{\alpha(1 + \xi^2)^p |\xi|^{2\beta}}{1 + \alpha(1 + \xi^2)^p |\xi|^{2\beta}} \right| \|\hat{g}(\cdot)\|_p. \end{aligned}$$

Since $h(\eta) := \eta(1 + \eta^2)^{-1}$, $\eta > 0$ has a maximum $1/2$,

$$\sup_{\xi \in R} \left| \frac{(1 + \xi^2)^{p/2} (i\xi)^\beta}{1 + \alpha(1 + \xi^2)^p |\xi|^{2\beta}} \right| \leq \frac{1}{2} (\sqrt{\alpha})^{-1} = \frac{1}{2} \frac{E}{\delta},$$

$$\sup_{\xi \in R} \left| \frac{\alpha(1 + \xi^2)^p |\xi|^{2\beta}}{1 + \alpha(1 + \xi^2)^p |\xi|^{2\beta}} \right| < 1.$$

Thus

$$\|\hat{f}_\delta^\alpha(\cdot) - \hat{f}(\cdot)\|_p \leq \frac{3}{2} E. \quad (3.6)$$

Hence according to (2.3) we have the error estimate

$$\|g_\delta^\alpha(\cdot) - g(\cdot)\| \leq \delta + \frac{3}{2} E^{\frac{\beta}{p+\beta}} \delta^{\frac{p}{p+\beta}}. \quad (3.7)$$

Now summarize what we have.

Theorem 3.1. *Suppose that $g(x)$ is the exact solution for problem (1.1) with exact data $f(x)$, (3.5) is the Tikhonov regularization solution with noisy data $f_\delta(x)$, (2.1) and (3.1) hold, then we have*

$$\|g_\delta^\alpha(\cdot) - g(\cdot)\| \leq \delta + \frac{3}{2} E^{\frac{\beta}{p+\beta}} \delta^{\frac{p}{p+\beta}}. \quad (3.8)$$

Remark 3.1. *Now we seek a solution in frequency domain for problem (1.1) in $L^2[-\xi_{\max}, \xi_{\max}]$ where $\xi_{\max}$ is a positive number which plays a role of regularization parameter. This is the so-called Fourier cut-off method. Similar to Tikhonov regularization, we can easily obtain the error estimate for Fourier cut-off method.*

4 Concluding remark

The issue of conditional stability on inverse ill-posed problems is very important. In this paper, we give an 'optimal' stability estimate. Based on the obtained estimate, the error estimate between Tikhonov regularization solution and the exact solution can be derived.

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